On approximating non-Archimedean Julia sets

joint work with Jean-Yves Briend

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Complex and $p$-adic Dynamics at ICERM
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Outline

1. Introduction
2. Non-Archimedean Julia set
3. Polynomials
4. Question
\[ K : \text{a field complete with respect to absolute value} \ | \cdot| \]

\[ \varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \text{ of degree } d \geq 2 \text{ over } K. \]

\[ J_\varphi(K) = \text{the } K\text{-rational Julia set of } \varphi \]

\[ = \text{the subset of points in } \mathbb{P}^1(K) \text{ where } \{\varphi^n\}_{n \geq 1} \]

is not equicontinuous.

**Problem**

To compute/visualize \( J_\varphi(K) \).

In general this is a difficult problem in the case where \( K = \mathbb{C} \).

**Theorem (Braverman-Yampolsky)**

There exist a complex number \( c \) such that the (complex) Julia set of the quadratic polynomial \( \varphi_c(z) = z^2 + c \) is not computable.
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Notation for non-Archimedean field

\( K \) a discretely valued field,
\( \nu: K^* \to \mathbb{Z} \) valuation on \( K \),
\( |x| = a^{-\nu(x)} \) for some \( a > 1 \),
\( \mathcal{O}_K \) the ring of integers of \( K \),
\( \pi \) a uniformizer such that \( \mathcal{M}_K = \pi \mathcal{O}_K \),
\( \tilde{K} = \mathcal{O}_K/\mathcal{M}_K \) assumed to be algebraically closed,
\( p = \text{Char}(\tilde{K}) \geq 0 \),
\( \mathcal{C}_\nu \) completion of an algebraic closure of \( K \),
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Write $\varphi(x, y) = [f(x, y), g(x, y)]$ with $f, g \in \mathcal{O}_K[x, y]$, homogeneous of degree $d$ with at least one coefficient being a unit. Set $\widetilde{\varphi} = [\widetilde{f}, \widetilde{g}]$.

**Good reduction**: $\varphi$ is said to have good reduction (over $\mathbb{C}_v$) if there exists a $\gamma \in \text{PGL}(2, \mathbb{C}_v)$ such that

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\varphi^\gamma(z) = (\gamma^{-1} \circ \varphi \circ \gamma)(z) = \frac{f(z)}{g(z)}, \quad f, g \in \hat{\mathcal{O}}_v[z]
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Although it’s not explicitly stated, P. Morton and J. Silverman’s work shows that:

**Theorem (Morton-Silverman)**

If $\varphi$ has good reduction over $\mathbb{C}_v$ then $J_\varphi = J_\varphi(\mathbb{C}_v)$ is empty.

**Remark (Properties of Julia set)**

1. $J_\varphi \subset \overline{\bigcup_m \text{Per}_m(\varphi)}$ (closure in $\mathbb{P}^1(\mathbb{C}_p)$).
2. $J_\varphi$ may not be compact in $\mathbb{P}^1(\mathbb{C}_p)$.
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Detecting the Julia set

• \( \mathcal{J}_\varphi(K) \subset \mathbb{P}^1(K) \subset \mathbb{P}^1_{\text{Berk}} \) (the Berkovich projective line).

We would like to compute \( \mathcal{J}_\varphi(K) \) as a subtree of \( \mathbb{P}^1_{\text{Berk}} \).

Julia set and Indeterminacies:

- \( X \) a smooth separated scheme of finite type over \( \mathcal{O}_K \), satisfying
  (i) the generic fiber \( X_\eta \simeq \mathbb{P}^1_K \); and
  (ii) \( \mathbb{P}^1(K) \simeq X(\mathcal{O}_K) \).

In this talk, we call such an \( X \) a model of \( \mathbb{P}^1_K \).

• Let \( \phi \) denote the extension of \( \varphi \) on \( X \). Then, in general we get a rational map

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\[ \overline{Q} = \text{the closure of } Q \text{ in } X. \]
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**Theorem**

Assume that $J_\phi(K)$ is non-empty and let $Q \in J_\phi(K)$. Then
\[ \{ \phi^n(Q) \mid n \geq 1 \} \text{ has non-empty intersection with the set of indeterminacies of } \phi. \]

**Remark**

(1) If there is a model $X$ of $\mathbb{P}^1_K$ such that the extension $\phi$ is a morphism on $X$, then $J_\phi(K)$ is empty.

(2) Such a model is called a weak Néron model for the pair $(\mathbb{P}^1_K, \phi)$ by Call and Silverman.
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(2) Such a model is called a weak Néron model for the pair \((\mathbb{P}^1_K, \varphi)\) by Call and Silverman.
Detecting the Julia set

Let $\tilde{X}$ denote the special fiber of $X$ and let $Q \in \mathbb{P}^1(K)$. 
$\overline{Q}$ = the closure of $Q$ in $X$. 
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Assume that $J_\varphi(K)$ is non-empty and let $Q \in J_\varphi(K)$. Then 
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An algorithm

Assume that we are given a model $X$ of $\mathbb{P}^1$ and an extension of $\varphi$

$$\phi : X \rightarrow X.$$ 

By blowing up the indeterminacies of $\phi$ we can find a model $X'$ and a birational morphism $\tau : X' \rightarrow X$ such that $\phi$ is lifted to a morphism

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such that we have a (triangle) diagram ! Repeat the process, we obtain a sequence of models $X_0 \xleftarrow{T_0} X_1 \xleftarrow{T_1} X_2 \leftarrow \cdots X_n \leftarrow \cdots$

Let $T_i$ be the dual graph of $\tilde{X}_i$. Then, we have

$$T_0 \rightarrow T_1 \rightarrow T_2 \rightarrow \cdots$$

Put $T_\varphi := \lim \rightarrow T_i \hookrightarrow \mathbb{P}^1_{\text{Berk}}$. Then, $J_\varphi(K) \simeq \partial T_\varphi$. 
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Liang-Chung Hsia, NTNU
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**Example**

(1) \( \phi(z) = f(z)/p \) where \( f(z) \in \mathbb{Z}_p[z] \) monic and

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f(z) \equiv z^p - z \pmod{p}
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Then, \( J_\phi = \mathbb{Z}_p \).

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Density question: Is $\mathcal{J}_\varphi$ the closure of repelling periodic points? Partial results under some conditions have been obtained.

Theorem (J.-P. Bézivin)

If $\varphi$ has at least one repelling periodic point, then $\mathcal{J}_\varphi$ is the closure of repelling periodic points.

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If the Lyapunov exponent $L(\varphi)$ is positive, then $\mathcal{J}_\varphi$ is the closure of repelling periodic points.

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The converse statement of Morton-Silverman Theorem does not hold in general.

Example

(1). Lattès family Let $E$ be an elliptic curve over $K$ and consider diagram:

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\begin{array}{ccc}
E & \xrightarrow{[m]} & E \\
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$\Rightarrow J_\varphi = \emptyset$.

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Example

(2). (Favre and Rivera-Letelier) Let $k \geq 2$ and $d_1, \ldots, d_k > 1$ be integers. Let $a_2, \ldots, a_k \in \mathbb{C}_v^*$ such that $|a_k| > \cdots > |a_2| > 0$. Set $\delta_1 = d_1, \delta_j = d_j + d_{j-1}$ for $j = 2, \ldots, k$ and

$$
\varphi(z) = z^{d_1} \prod_{j=2}^{k} \left(1 + (a_j z)^{\delta_j}\right)^{(-1)^j}.
$$

If $\sum d_j^{-1} \leq 1$ then there exist $a_2, \ldots, a_k$ such that $\mathcal{J}_\varphi = \emptyset$ and $\varphi$ does not have good reduction over $\mathbb{C}_v$. 

We restrict to the case $\varphi(z) \in K[z]$.

- $J_\varphi(K)$ is a compact subset of $\mathbb{P}^1(K)$.

**Goal:** Look for an effective algorithm to determine whether $J_\varphi(K)$ is empty or not.
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Quadratic family

Low degrees \( d = 2, 3 \):

For the quadratic family, the situation is much simpler than the classical case \((K = \mathbb{C})\). Let \( \varphi_c(z) = z^2 + c \) and \( J_c(K) = J_{\varphi_c}(K) \).

**Theorem (Benedetto-Briend-Perdry)**

\[ J_c(K) \neq \emptyset \text{ if and only if one of the following conditions holds.} \]

\begin{enumerate}
  \item \( p \neq 2 \): \( v(c) = -2k < 0 \);
  \item \( p = 2 \): \( v(4c) < 0 \) and \( 1 - 4c \) is a square in \( K \).
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In this case \((J_c(K) \neq \emptyset)\), we have \( J_c(K) = J_c \) and the dynamics of \( \varphi \) on \( J_c(K) \) is topologically conjugated to the full (one-sided) 2-shift.

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Introduction
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Polynomials
Question

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2. \( p = 2 \): \( v(4c) < 0 \) and \( 1 - 4c \) is a square in \( K \).

In this case \((\mathcal{J}_c(K) \neq \emptyset)\), we have \( \mathcal{J}_c(K) = \mathcal{J}_c \) and the dynamics of \( \varphi \) on \( \mathcal{J}_c(K) \) is topologically conjugated to the full (one-sided) 2-shift.

- We can rephrase the theorem as the following

\( \mathcal{J}_c(K) \neq \emptyset \) if and only if the two (finite) fixed points of \( \varphi_c \) are \( K \)-rational and repelling.
Introduction
Non-Archimedean Julia set
Polynomials
Question

Quadratic family

Low degrees \(d = 2, 3\):

For the quadratic family, the situation is much simpler than the classical case \((K = \mathbb{C})\). Let \(\varphi_c(z) = z^2 + c\) and \(J_c(K) = J_{\varphi_c}(K)\).

Theorem (Benedetto-Briend-Perdry)

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Cubic polynomials

For cubic polynomials criterion for the existence of $K$-rational Julia set is similar to the quadratic family.

**Theorem (Briend-Hsia)**

Let $\varphi$ be a cubic polynomial. Then the $K$-rational Julia set $J_\varphi(K) \neq \emptyset$ if and only if one of the fixed point of $\varphi$ is $K$-rational and repelling.
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Remarks

- (B.-H.) The same criterion holds for the family of quadratic rational maps.
- Proofs show that $J_\varphi = \emptyset \implies \varphi$ has good reduction over $\mathbb{C}_v$.
- For $d = 2, 3$, the criterion for determining the existence of $J_\varphi (J_\varphi (K))$ is effective.
- Remark made by Silverman: the above criterion for rational maps follows from his theorem on the $\mathbb{Z}$-structure of the moduli space $\mathcal{M}_d$ and that $\mathcal{M}_2$ is isomorphic to $\mathbb{A}^2_{\mathbb{Z}}$ as schemes over $\mathbb{Z}$. 
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The following example of Benedetto shows that for polynomial maps the converse of Morton-Silverman’s result does not hold either.

Example (R. Benedetto)

(1) Let the residue characteristic $p$ be odd. Let $a \in \mathbb{C}_v$ satisfy $-p/(2p+2) \leq v(a) < 0$. Let $\varphi(z) = z^2(z - a)^p$. Then, $J_\varphi = \emptyset$ and $\varphi$ does not have good reduction over $\mathbb{C}_v$.

(2) For $p = 2$, the following example shares the same property as in (1)

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Higher degree polynomials

For polynomial with $\deg \varphi \geq 4$, the above criterion for $d = 2, 3$ does not hold in general.
Let $(p, d) \not\in \{(2, 4), (2, 5), (2, 7), (3, 5)\}$.
Write $d = e_0 + e_1$ or $e_0 + e_1 + e_2$ such that $e_i \geq 2$ and $p \nmid e_i$ (if $p = 0$, this condition is empty).
Let $n = \text{lcm}(\{e_i\})$.

Example $(d = e_0 + e_1)$

Let

$$\varphi(z) = \frac{1}{\pi^n} z^{e_0} (z - 1)^{e_1} + z.$$ 

Then, $\text{Fix}(\varphi) = \{0, 1, \infty\}$ and non-repelling.
$\mathcal{J}_\varphi(K) \neq \emptyset$ and the minimal period of the repelling periodic points is 2.
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Consider $\sigma : \mathcal{P}_d \to \mathbb{A}^N$ where $\mathcal{P}_d$ denotes the moduli space of polynomials of degree $d$. Let $[\varphi] \in \mathcal{P}_d(\mathbb{C}_v)$.

If $\mathcal{J}_\varphi = \emptyset$ then all periodic points are non-repelling.

Hence $\sigma([\varphi]) \in \mathbb{A}^N(\widehat{\mathcal{O}_K})$.

Question

Is it true that $\sigma^{-1}(\mathbb{A}^N(\widehat{\mathcal{O}_K}))$ consist of all polynomials $\varphi$ with empty Julia set?
Integral points

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We believe the following is true.

**Conjecture**

Let $\varphi \in K[z]$ be of degree $d$. There exist a constant $N = N(p, d)$ such that $J_\varphi = \emptyset$ if and only if all the periodic points of period $r$ with $1 \leq r \leq N$ are non-repelling.
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